A PERSPECTIVE FOR EXAMINING THE LINK BETWEEN PROBLEM POSING AND PROBLEM SOLVING

Jinfa Cai Stephen Hwang
University of Delaware

In a previous study, we posited a link between Chinese sixth grade students' problem solving and problem posing based on a pattern-formation strategy (Cai & Hwang, 2002). A similar parallel structure between problem solving and problem posing did not obtain for the U.S. sixth graders in the study. The present study attempts to locate this type of parallel structure by analyzing a broader sample of U.S. students. The results of this study show that U.S. seventh graders are much more likely than sixth graders to use abstract strategies. The findings appear to support a relationship between the use of abstract problem solving strategies and the tendency to pose extension problems (problems that go beyond the given information) for the seventh graders.

INTRODUCTION

Problem posing lies at the heart of mathematical research and scientific investigation. Indeed, in scientific inquiry the process of formulating a problem well can be more significant than the discovery of a solution to the problem (Einstein & Infeld, 1938). In mathematics education, there is a broad consensus that students should have the opportunity to develop their mathematical problem posing abilities (Brown & Walter, 1990; NCTM, 2000). It has been suggested that such activities not only help to lessen students' anxiety and foster a more positive disposition towards mathematics, but they may also enrich and improve students' understanding and problem solving (Brown & Walter, 1990; NCTM, 2000; Silver, 1994).

Given the importance of problem-posing activities in school mathematics, some researchers have started to investigate various aspects of problem-posing processes (e.g., Silver, 1994). One important direction for such investigation has been to examine the link between problem posing and problem solving (e.g., Cai & Hwang, 2002; English, 1997; Silver & Cai, 1996). Kilpatrick (1987) provided a theoretical argument that the quality of the problems subjects pose might serve as an index of how well they can solve problems. Certainly, effective problem solving strategies can often involve the posing of related or subsidiary problems (Polya, 1957). In addition, several researchers have conducted empirical studies to examine the link between problem posing and problem solving. For example, Silver and Cai (1996) analyzed the responses of more than 500 middle school students to a task asking them to pose three questions based on a driving situation. The students' posed problems were analyzed by type, solvability, and complexity. Silver and Cai used eight open-ended tasks to measure the students' mathematical problem-solving performance. They found that problem-solving performance was highly correlated with problem-posing performance. Compared with less successful problem solvers good problem solvers generated more, and more mathematically complex, problems. However, in this study the problem solving tasks and problem posing tasks were not embedded in parallel mathematical contexts.

In a recent cross-national comparative study, Cai and Hwang (2002) examined the relationships between sixth grade students' problem posing and problem solving using tasks with identical mathematical structures. They found differential relationships between problem posing and problem solving for U.S. and Chinese students. There was a much stronger link between problem solving and problem posing for the Chinese sample than there was for the U.S. sample. In addition, they found a parallel between the Chinese students' problem solving strategies and the sequences of problems they posed, while no such parallel could be identified for the U.S. sample. Cai and Hwang speculated that the differential relationships between problem posing and problem solving for U.S. and Chinese students might be due to the disparities in the U.S. and Chinese students' problem-solving strategies. While the Chinese students were more likely to use abstract strategies, the U.S. students almost exclusively used concrete strategies and drawing representations. This relatively consistent lack of abstract strategy use made it impossible to identify a similar link between problem solving and problem posing for the U.S. sixth graders.

Only sixth grade U.S. and Chinese students were included in the above-mentioned study. From a developmental perspective, it is probable that older students would be more likely to use abstract problem solving strategies. The main purpose of the present study is to examine both U.S. sixth and seventh grade students' mathematical problem solving and problem posing in the hope of identifying a link between the two. If, indeed, the seventh graders are much more likely to use abstract problem solving strategies than the sixth graders, one would hope to see a similar link between problem solving and problem posing as was observed in the Chinese sample in Cai and Hwang (2002).

METHOD

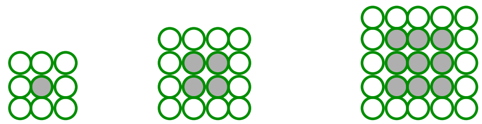
Subjects

A total of 98 sixth graders (42 girls and 56 boys) and 109 seventh graders (52 girls and 57 boys) participated in the study. These students were selected from four public schools in suburban Pittsburgh. Although the sample is not atypical for the Pittsburgh area, the expectations for students in these schools are high, with each of the communities proud of the percentage of their students who go on to higher education. The majority of the students come from middle-class families; only a small proportion (about 3%) would qualify for reduced-price lunches.

Tasks and administration

Three pairs of problem-solving and problem-posing tasks were administered to each student. Figure 1 shows the two pairs of tasks for which data are reported in this paper. For each of the problem posing tasks, students were asked to pose one easy problem, one moderately difficult problem, and one difficult problem. For each of the problem-solving tasks, students were asked to answer several questions based on the given pattern. The problem-solving tasks were collected in one booklet and the problem-posing tasks in another. Students were given 20 minutes to complete the tasks in the problem-posing booklet. The following day, students were given 40 minutes to complete the three problem-solving tasks. All data were collected via students' written responses.

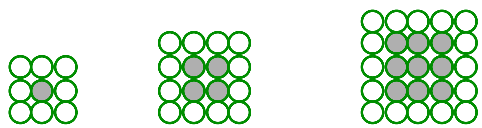
Dots Problem-Solving: Look at the figures below.



(Figure 1) (Figure 2) (Figure 3)

1. Draw the 4th figure.
2. How many black dots are there in the 6th figure? Explain how you found your answer
3. How many white dots are there in the 6th figure? Explain how you found your answer.
4. Figure 1 has 8 white dots. Figure 3 has 16 white dots. If a figure has 44 white dots, which figure is this? Explain how you found your answer.

Dots Problem-Posing: Mr. Miller drew the following figures in a pattern, as shown below.



(Figure 1) (Figure 2) (Figure 3)

For his student's homework, he wanted to make up three problems BASED ON THE ABOVE SITUATION: an easy problem, a moderate problem, and a difficult problem. These problems can be solved using the information in the situation. *Help Mr. Miller make up the three problems.*

Doorbell-Solving: Sally is having a party, the first time the doorbell rings, 1 guest enters.
The second time the doorbell rings, 3 guests enter.
The third time the doorbell rings, 5 guests enter.
The fourth time the doorbell rings, 7 guests enter.

Keep on going in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring.

1. How many guests will enter on the 10th ring? Explain how you found your answer.
2. In the space below, write a rule or describe in words how to find the number of guests that entered on each ring.
3. 99 guests entered on one of the rings. What ring was it? Explain or show how you found your answer.

Doorbell-Posing: Sally is having a party, the first time the doorbell rings, 1 guest enters.
The second time the doorbell rings, 3 guests enter.
The third time the doorbell rings, 5 guests enter.
The fourth time the doorbell rings, 7 guests enter.Keep on going in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring. For his student's homework, Mr. Miller wanted to make up three problems BASED ON THE ABOVE SITUATION: an easy problem, a moderate problem, and a difficult problem. These problems can be solved using the information in the situation. *Help Mr. Miller make up the three problems.*

Figure 1: Two pairs of problem-solving and problem-posing tasks

Data coding and inter-rater reliability

Student responses were coded using the schema in Cai and Hwang (2002). Each response to a problem-solving task was coded for three factors: correctness of answer, mode of representation, and type of solution strategy. Details about solution strategies and representations are described in the results section. Student responses to the problem-posing task were coded along two dimensions. The posed problems were first classified into extension problems, non-extension problems, or others. A problem is considered an extension problem if it asks about the pattern beyond the first several given figures or terms. A non-extension problem restricts itself solely to the first several given figures or terms in a pattern. After this initial coding, each extension or non-extension problem was further categorized according to the nature of the problem. To ensure high reliability in the data analysis, two raters independently coded at least 10% of the student responses from each sample. The inter-rater agreements were 91% to 100% for coding correctness of answers, solution strategies, and solution representations in problem solving tasks. The inter-rater agreements were 84% to 92% for coding responses to the problem posing tasks.

RESULTS

Problem solving

Table 1 shows the percentages of sixth and seventh grade students correctly answering each of the four Dots questions and two Doorbell questions. Overall, the seventh graders have a significantly higher mean score for the six questions than the sixth graders (means of 3.8 and 2.8, respectively; $t = 4.505$, $p < .001$). As Table 1 shows, the results are essentially consistent across the two tasks. As might be expected from the increasingly abstract nature of the problems in each situation, the students in each grade experienced more difficulty as they worked through the sequence of problems in each situation. Comparing the performance of students across grade levels, it is clear that the seventh graders generally outperformed the sixth graders on all problems.

Grade	Dots Q1	Dots Q2	Dots Q3	Dots Q4	Doorbell Q1	Doorbell Q3
6 (n=98)	76.5	50.0	35.7	27.6	70.4	23.5
7 (n=109)	87.2	75.2	50.5	39.4	83.5	41.3
p	.05	.00	.03	.07	.03	.01

Table 1: Students' success rates on each of the Dots and Doorbell questions

To examine students' solution strategies, we focused on the last Dots question and the last Doorbell question. These questions were the most amenable to the use of abstract problem solving strategies. Each strategy was coded as concrete, semi-abstract, or abstract. Concrete strategies involve making lists, drawing pictures, or guessing and checking. Semi-abstract strategies consist of multiple computational steps without a recognition of the overall pattern. For example, a semi-abstract strategy for finding the ring number at which 99 guests entered might look like: $99 - 9 = 90$; $90/2 = 45$; $45 + 5 = 50$. Abstract strategies were based on some recognition of the general pattern governing the problem situation. For example, for the Dots questions, a student who realized that

the number of white dots in the nth figure was $(n + 2)^2 - n^2$ was coded as using an abstract strategy. Strategies that were fundamentally incapable of producing a correct answer were coded as unfeasible. Table 2 shows the percentage distributions of students in each grade using concrete, semi-abstract, and abstract strategies.

	Dots Question 4				Doorbell Question 3			
Grade	Concrete	Semi	Abstract	Other	Concrete	Semi	Abstract	Other
6 (n=98)	62.2	5.1	0.0	32.7	35.7	7.1	5.1	52.0
7 (n=109)	52.3	10.1	0.0	37.6	29.4	13.8	11.0	45.9

Table 2: Percentages of students using each of the strategies

As expected, abstract strategy use increased with grade level. The seventh grade students tended to use more abstract strategies than the sixth graders. However, this trend is more evident in the Doorbell question than in the Dots question. This is primarily due to the fact that no student in either grade chose an abstract strategy for the Dots question. There was, however, an increase in the use of semi-abstract strategies for the seventh graders.

Comparing the abstractness of strategies with students' problem-solving success, it appears that those sixth and seventh grade students who used more abstract strategies tended to have a higher success rate. In particular, students who used an abstract strategy for Doorbell question 3 had a success rate of 85%, but those students who used concrete strategies only had a success rate of 53%. Because no student used an abstract strategy for the last Dots question, there is no data to make a similar comparison.

Problem posing

As noted above, each posed problem was coded as an extension problem or a non-extension problem. In general, an extension problem is a problem concerning the pattern beyond the given figures. However, some problems were phrased so generally as to make it difficult or impossible for a solver to know how to answer them (e.g., "What's the pattern?"). Based on our previous work, we chose to exclude these problems from the category of extension problems in further analyses.

Looking from the sixth to the seventh graders, there is a definite trend in extension problem posing. In both the Dots and Doorbell situations, the seventh graders appear to pose more extension problems. The mean number of extension problems posed by sixth graders is 0.48 for the Dots situation and 1.58 for the Doorbell situation. In contrast, the mean number of extension problems posed by seventh graders is 1.03 for the Dots situation and 1.83 for the Doorbell situation.

Table 3 shows the percentage of students in each grade posing zero, one, two, or three extension problems for both Dots and Doorbell situations. While less than 15% of the sixth graders generated at least two extension problems for the Dots Situation, about 37% of the seventh graders did so ($z = 3.86$, $p < .01$). A similar, but weaker version of this pattern holds for the Doorbell data. Overall, the students appear to have produced more extension problems for the Doorbell situation than for the Dots situation.

	Dots Situation				Doorbell Situation			
Grade	Zero	One	Two	Three	Zero	One	Two	Three
6 (n=98)	69.4	17.3	9.2	4.1	28.6	15.3	25.5	30.6
7 (n=109)	48.6	14.7	22.0	14.7	20.2	15.6	25.7	38.5

Table 3: Percentages of students posing zero, one, two, and three problems

Beyond the division into extension and non-extension problems, the posed problems were coded into specific categories based on the types of questions being asked. The posed problems for the Dots situation were classified as: (1) Count dots in one figure, (2) Count dots in multiple figures, (3) Compare number of dots in figure(s), (4) Draw a figure, (5) Specific rule-based problems, and (6) General rule-based problems. The posed problems for the Doorbell situation were classified into: (1) Number of guests on a ring, (2) Ring number for some number of entering guests, (3) Total number of guests for several rings, (4) Total number of rings for some total number of entered guests, and (5) Specific rule-based problems.

Regarding the type of problems posed, there is little difference between the sixth and seventh graders in either situation. Looking across all the posed problems related to the Dots situation in both sixth and seventh grades, the most common types were the general rule-based problems and extension problems asking for the number of dots (black, white, or both) in a single figure. In addition, there were a considerable number of extension problems that asked for a drawing of a figure. In the Doorbell situation, the most common posed problem type was an extension problem asking for the number of guests entering at a given ring number. A substantial number of the posed problems were coded as irrelevant or missing. Of the remaining cases, the most common were extension problems asking for the total number of guests that had entered at a given ring number.

Relationships between problem solving and problem posing

Since the students did not use any abstract strategies for the Dots problem-solving task, this part of the analysis is limited to the Doorbell problem-solving task. For the seventh graders, students who posed at least two extension problems tended to use problem-solving strategies that were more abstract. Specifically, over 30% of the students who posed at least two extension problems used abstract strategies, but only about 14% of the students who posed fewer than two extension problems did so. For the sixth graders, this pattern does not appear to hold. This seems to be due to the fact that very few sixth graders chose to use abstract strategies.

Problem-solving success appears to be related to the tendency to pose extension problems across the two grade levels and two problem situations. Table 4 shows the relationship between problem solving performance and the number of extension problems posed. Indeed, the sixth graders posing at least two extension problems outscored their counterparts posing fewer than two on every problem except the first Dots problem, often by more than ten percentage points. In the case of the seventh graders, all the Dots and Doorbell questions follow this pattern.

	Dots Questions				Doorbell Questions	
	Q1	Q2	Q3	Q4	Q1	Q2
6th Graders						
At Least 2 Extension	69.2	61.6	53.8	38.5	92.3	30.8
Fewer Than 2 Extension	77.6	48.2	32.9	25.9	67.1	22.4
7th Graders						
At Least 2 Extension	92.5	85.0	52.5	40.0	87.5	50.0
Fewer Than 2 Extension	84.1	69.6	49.3	39.1	81.2	36.2

Table 4: Problem solving success vs. Number of extension problems posed

DISCUSSION

In our previous study, we posited a link between Chinese sixth grade students' problem solving and problem posing based on the notion of a pattern-formation strategy (Cai & Hwang, 2002). The Chinese students tended to solve problems using abstract strategies, finding and applying formal patterns. This tendency toward a pattern-formation strategy found a parallel in the problem posing results. A common sequence of problems posed by the Chinese students began with a problem that could be interpreted as data collection (e.g., counting the number of dots in multiple given figures). This was followed by a problem that resembled a data-analysis and trend-seeking strategy (e.g., comparing the number of dots across multiple given figures). The final problem was often of an extension type, as if testing or making use of a proposed pattern. A similar parallel structure between problem solving and problem posing thinking did not obtain in the U.S. sixth graders in the study.

The present study attempts to locate such a parallel by analyzing a broader sample of U.S. students. One way to conceptualize a link between problem solving and problem posing activities lies in the realization that complex problem solving processes often involve the generation and solution of subsidiary problems (Polya, 1957). Because extension problems are defined as involving an extrapolation from given data to unknown situations, it seems reasonable to believe that the ability to pose such problems would be associated with more robust problem solving abilities. More specifically, one might expect that abstract problem-solving strategies would be most benefited by a propensity for posing extension problems, since abstract strategies depend on the ability to identify and make use of the patterns that define a situation. The results of this study show that U.S. seventh graders are much more likely than the sixth graders to use abstract strategies. In addition, the findings appear to support a relationship between abstract problem solving strategy use and the tendency to pose extension problems for the seventh graders.

Both this study as well as the previous study (Cai & Hwang, 2002) provide one perspective from which to examine the link between students' problem solving and problem posing. In particular, they suggest the feasibility of studying this link by examining the relationship between types of problem solving strategies and the tendency

to pose extension problems. This perspective should have direct instructional implications. Researchers (e.g., Silver, 1994) have suggested that student-posed problems are more likely to connect mathematics to students' own interests, something that is often not the case with traditional textbook problems. Thus, encouraging students to generate extension problems may not only foster positive attitudes toward mathematics and greater understanding of problem situations, but it may also help students develop more advanced problem-solving strategies.

Acknowledgements

Preparation of this paper is supported, in part, by a grant from the Spencer Foundation. Any opinions expressed herein are those of the authors and do not necessarily represent the views of the Spencer Foundation. We are very grateful for the assistance of Marcia Seeley for data collection. Thanks also go to Yi Long, Jennifer Nsian-Kumi, Tao Wang, and Sylvia Yuan for data coding and analysis.

References

Brown, S. I., & Walter, M. I. (1990). *The art of problem posing* (2nd ed.). Hillsdale, NJ: Erlbaum.

Cai, J., & Hwang, S. (2002). Generalized and generative thinking in U.S. and Chinese students'mathematical problem solving and problem posing. *Journal of Mathematical Behavior, 21*, 401-421.

Einstein, A., & Infeld, L. (1938). *The evolution of physics*. New York: Simon & Schuster.

English, L. D. (1997). The development of fifth-grade children's problem-posing abilities. *Educational Studies in Mathematics, 34*, 183-217.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Hillsdale, NJ: Erlbaum.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Polya, G. (1957). *How to solve it* (2nd ed.). Princeton, NJ: Princeton University Press.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19-28.

Silver, E. A. & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*, 521-539.